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Countable dimensionality and dimension raising cell-like maps

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Abstract

We show that countable dimensionality is not preserved under hereditary shape equivalences between complete spaces if such maps can make "arbitrarily large dimension jumps" inside the class of countably dimensional compacta. © 1997 Elsevier Science B.V.

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We assume throughout this paper that every space is separable metric. We use *map* as an abbreviation for continuous mapping. A map is called *proper* if the preimage of every compactum is compact or, equivalently, if the map is closed and has compact fibres. Let us recall that a proper map f from X onto Y is a *cell-like* map if, for every $y \in Y$, $f^{-1}(y)$ is of trivial shape, i.e., $f^{-1}(y)$ is a cell-like set in X. We say that the proper surjection f is a *hereditary shape equivalence* if for every closed subset A in Y $f|f^{-1}(A): f^{-1}(A) \to A$ is a shape equivalence, i.e., for each ANR Z the map $f|f^{-1}(A)$ produces a one-to-one correspondence between the homotopy classes of $C(A, Z)$ and $C(f^{-1}(A), Z)$. A space X is *countably dimensional* if it is a countable union of finite-dimensional subspaces.

It is a well-known corollary of Alexandroff's Essential Mapping Theorem [1] that finite dimension is preserved under hereditary shape equivalences. The question whether

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transfinite dimension properties are preserved under hereditary shape equivalences is now solved with the exception of the countable dimensionality case (posed for compacta by Henderson, Kozlowski and Walsh at the problem session of the AMS meeting in Norman, 1983):

Problem. *If $h: X \to Y$ is a hereditary shape equivalence and X is countably dimensional must Y also be countably dimensional?*

It turns out that this problem is closely related to the behaviour of transfinite dimension under hereditary shape equivalence. Let us recall that there are natural extensions of small and large inductive dimensions over countably dimensional spaces (see, e.g., [8]). If α is a countable ordinal ($\alpha < \omega_1$), then $\operatorname{ind} X \leqslant \alpha$ if for every $x \in X$ and for every open set U, with $x \in U$, there exists an open set V such that $x \in V \subset U$ and $\operatorname{ind}(\partial V) < \alpha$; $\operatorname{Ind} X \leqslant \alpha$ if for every closed subset A of X and for every open set U, with $A \subset U$, there exists an open set V such that $A \subset V \subset U$ and $\operatorname{Ind}(\partial V) < \alpha$.

Using the fact that a complete space X is countably dimensional if and only if $\operatorname{ind} X$ exists and that a cell-like map with a countably dimensional range is a hereditary shape equivalence [2, Corollary 5.3] we define

$$\eta(X) = \sup\{\operatorname{ind} Y\colon\ Y \text{ is a countably dimensional cell-like image of } X\}.$$

It was proved in [5, Proposition 4.5] that if for every countably dimensional compactum X we have that $\eta(X)$ is countable, then hereditary shape equivalences between compacta preserve countable dimensionality. We announced in [5] and [6, Theorem 3.3] that the converse of this result is also valid. Unfortunately, our proof has a gap and we take this opportunity to retract the announcement (see also the remarks at the end of this note). However, we can prove the following weaker statement:

Theorem 1. *If hereditary shape equivalences between separable, complete metric spaces preserve countable dimensionality, then for every countable ordinal α we have*

$$\sup\{\eta(X)\colon\ X \text{ is a compactum with } \operatorname{ind} X \leqslant \alpha\} < \omega_1.$$

Since for compacta $\operatorname{ind} X \leqslant \operatorname{Ind} X \leqslant \omega \cdot \operatorname{ind} X$ (see [11]) this theorem remains valid if we replace ind by Ind in the definition of the function η. This theorem reduces the Problem to finding certain dimension raising maps for ind. In this direction the following result is available [4]:

Theorem 2. *There exists a cell-like map from an ω-dimensional compact AR onto an $(\omega + 1)$-dimensional compact AR.*

This shows that the transfinite dimension functions ind and Ind are not preserved under hereditary shape equivalences. Theorems 1 and 2 suggest a strategy of attacking Problem. We summarize this strategy with the following conjectures.

Conjecture 3. *For every countable ordinal α there is a hereditary shape equivalence from an ω-dimensional compactum onto a compactum with* $\operatorname{ind} > \alpha$.

This conjecture combines with Theorem 1 to

Conjecture 4. *Countable dimensionality is not preserved under hereditary shape equivalences between complete spaces.*

For a more complete discussion of the behaviour of certain "dimensionality properties" of infinite-dimensional spaces under hereditary shape equivalences see [3–6,12,13].

We need some definitions. A proper surjection $f:X \to Y$ is called a (*homotopic*) *near homeomorphism* if for every open covering $\mathcal{V}$ of Y there is a homeomorphism $h:X \to Y$ such that f and h are $\mathcal{V}$-close ($\mathcal{V}$-homotopic). A proper surjection $f:X \to Y$ is called (*homotopically*) *shrinkable* if for every pair of open coverings $\mathcal{U}$ and $\mathcal{V}$ of X respectively Y there is a homeomorphism $h:X \to X$ such that h and the identity 1_X are $f^{-1}[\mathcal{V}]$-close ($f^{-1}[\mathcal{V}]$-homotopic) and the collection of fibres of $f \circ h$ refines $\mathcal{U}$.

The next two propositions state facts that are well-known for ANRs. We have included proofs since we need their application in the setting of fibrations over the irrationals. Consider the following homotopic version of Bing's Shrinking Criterion (cf. [7,15]):

Proposition 5. *A map between complete spaces is a homotopic near homeomorphism if and only if it is homotopically shrinkable.*

Proof. The proof is analogous to that of regular shrinking; we concentrate on the homotopy aspect.

Assume that $f:X \to Y$ is a homotopic near homeomorphism and let $\mathcal{U}$ and $\mathcal{V}$ be open covers of X respectively Y. Let $G_t:X \to Y$ be a $\mathcal{V}$-limited homotopy such that $G_0 = f$ and G_1 is a homeomorphism. Now select a homotopy $H_t:X \to Y$ that is both $\mathcal{V}$-limited and $G_1[\mathcal{U}]$-limited and with the properties $H_0 = f$ and that H_1 is a homeomorphism. Consider the autohomeomorphism $h = H_1^{-1} \circ G_1$ of X and note that the $H_1^{-1}[\mathcal{V}]$-limited homotopy $H_1^{-1} \circ G_t$ connects h with $H_1^{-1} \circ f$. Since H_1 and f are $\mathcal{V}$-close we have that $H_1^{-1}[\mathcal{V}]$ refines $f^{-1}[\mathrm{st}(\mathcal{V})]$. So h and $H_1^{-1} \circ f$ are $f^{-1}[\mathrm{st}(\mathcal{V})]$-homotopic. Note that the homotopy $H_1^{-1} \circ H_t$ connects $H_1^{-1} \circ f$ with 1_X and that it is also $f^{-1}[\mathrm{st}(\mathcal{V})]$-limited for the same reason. So h and 1_X are $f^{-1}[\mathrm{st}(\mathrm{st}(\mathcal{V}))]$-homotopic. Consider now a fibre of $f \circ h$, $G_1^{-1}(H_1(f^{-1}(y)))$. Since H_1 and f are $G_1[\mathcal{U}]$-close, we have $H_1(f^{-1}(y)) \subset \mathrm{st}(y, G_1[\mathcal{U}])$. Consequently, $G_1^{-1}(h_1(f^{-1}(y))) \subset \mathrm{st}(G_1^{-1}(y), \mathcal{U})$ and hence the fibres of $f \circ h$ form a refinement of $\mathrm{st}(\mathcal{U})$. This proves that f is homotopically shrinkable.

Assume now that $f:X \to Y$ is a homotopically shrinkable map between complete spaces. Select arbitrary complete metrics d and ρ on X respectively Y. We construct a sequence $g_0, g_1, \ldots$ of homotopically shrinkable maps from X to Y with the property that every fibre of g_n has d-diameter less than 2^{-n} for $n \geqslant 1$. Put $g_0 = f$ and assume that g_n has been constructed. Define the collection $\mathcal{W}$ of open subsets of Y by

$$\mathcal{W} = \{W \subset Y:\ W \text{ open}, \rho\text{-diam}(W) < 2^{-n-1},$$
$$\text{and if } n \geqslant 1 \text{ then } d\text{-diam}(g_n^{-1}(W)) < 2^{-n}\}.$$

Since g_n is proper $\mathcal{W}$ covers Y. Select a homeomorphism $h_n : X \to X$ such that every fibre of $g_{n+1} = g_n \circ h_n$ has d-diameter at most 2^{-n-1} and g_n and g_{n+1} are $\mathcal{W}$-homotopic. Since homotopic shrinkability is obviously topologically invariant we have that g_{n+1} is also homotopically shrinkable. Note that g_n and g_{n+1} are 2^{-n-1}-homotopic with respect to ρ. So we have a uniform Cauchy sequence of homotopies which we can paste together to form a 1-homotopy that connects f with $g = \lim_{n\to\infty} g_n$. The proof that g is a homeomorphism is identical to the proof for regular shrinking. □

A map $f : X \to Y$ is called a *fine homotopy equivalence* if for every open covering $\mathcal{V}$ of Y there is a map $g : Y \to X$ such that $f \circ g$ and 1_Y are $\mathcal{V}$-homotopic and $g \circ f$ and 1_X are $f^{-1}[\mathcal{V}]$-homotopic.

Proposition 6. *Every homotopic near homeomorphism is a fine homotopy equivalence.*

Proof. Assume that $f : X \to Y$ is a homotopic near homeomorphism and let $\mathcal{V}$ be an open covering of Y. Let $H_t : X \to Y$ be a $\mathcal{V}$-limited homotopy such that $H_0 = f$ and H_1 is a homeomorphism. Put $g = H_1^{-1}$. Note that the $\mathcal{V}$-limited homotopy $H_t \circ g$ connects $f \circ g$ with 1_Y, where as $g \circ H_t$ is $g[\mathcal{V}]$-limited and connects $g \circ f$ with 1_X. Since g^{-1} and f are $\mathcal{V}$-close we have that $g[\mathcal{V}]$ refines $f^{-1}[\mathrm{st}(\mathcal{V})]$. So $g \circ f$ and 1_X are $f^{-1}[\mathrm{st}(\mathcal{V})]$-homotopic. □

The following relations are well-known. Every proper surjective fine homotopy equivalence is a hereditary shape equivalence. For maps from the Hilbert cube Q onto itself all these concepts, (homotopic) near homeomorphism, (homotopically) shrinkable map, fine homotopy equivalence, hereditary shape equivalence, and cell-like map, are interchangeable.

The proof of Theorem 1 is preceded by two lemmas. Let $\mathbb{P}$ stand for the space of irrational numbers.

Lemma 7. *Let Z be a compactum and let f be a proper map of the product $\mathbb{P} \times Z$ onto a space E with $f^{-1}(f(\{p\} \times Z)) = \{p\} \times Z$ for all $p \in \mathbb{P}$. Then f is a hereditary shape equivalence if and only if $f|\{p\} \times Z$ is a hereditary shape equivalence for all $p \in \mathbb{P}$.*

Proof. The "only if" part is trivial.

Let $f : \mathbb{P} \times Z \to E$ be a proper surjection such that $f^{-1}(f(\{p\} \times Z)) = \{p\} \times Z$ and $f|\{p\} \times Z$ is a hereditary shape equivalence for all $p \in \mathbb{P}$. Note that $\mathbb{P} \times Z$ is completely metrizable and so is E because f is proper. Assume that Z is a Z-set in Q and define an upper semicontinuous cell-like decomposition

$$\mathcal{G} = \{f^{-1}(y):\ y \in E\} \cup \{\{x\}:\ x \in \mathbb{P} \times (Q \setminus Z)\}$$

of $\mathbb{P} \times Q$. The quotient map F of $\mathbb{P} \times Q$ onto $E' = \mathbb{P} \times Q/\mathcal{G}$ is a cell-like map such that $F^{-1}(F(\{p\} \times Q)) = \{p\} \times Q$. Define for each $p \in \mathbb{P}$, $Q_p = F(\{p\} \times Q)$ and $F_p : Q \to Q_p$ by $F_p(q) = F(p, q)$. Since $F_p|Z$ is a hereditary shape equivalence and the nondegeneracy set of F_p is contained in the closed set Z the map F_p is also a hereditary shape equivalence. According to [9] the image of an AR under a hereditary shape equivalence is an AR as well so Q_p is an AR. Since Z is a Z-set we may conclude that F_p is a near homeomorphism and Q_p is a Hilbert cube for all $p \in \mathbb{P}$ (see [16, Theorem 1.1]).

We will show that F is homotopically shrinkable. Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of $\mathbb{P} \times Q$ and E', respectively. Since F_p is homotopically shrinkable we have for each $p \in \mathbb{P}$ a homotopy $H_t^p : Q \to Q$ that is limited by $F_p^{-1}[\mathcal{V}]$ such that $H_0^p = 1_Q$, H_1^p is a homeomorphism of Q, and the collection of fibres of $F_p \circ H_1^p$ refines $\mathcal{U}$.

Consider the proper map $G_p = F \circ (1_{\mathbb{P}} \times H_1^p)$ and define for each $p \in \mathbb{P}$,

$$U_p = \bigcup \{G_p^{-1}(y)\colon\ G_p^{-1}(y) \subset U \text{ for some } U \in \mathcal{U}\}.$$

Since G_p is like F a closed map U_p is open in $\mathbb{P} \times Q$ and since

$$G_p^{-1}(y) = \{p\} \times \left(F_p \circ H_1^p\right)^{-1}(y)$$

for $y \in Q_p$, we have $\{p\} \times Q \subset U_p$. Define for each $p \in \mathbb{P}$ the subset V_p of $\mathbb{P} \times Q$ by

$$V_p = \left\{(r,q) \in \mathbb{P} \times Q\colon\ \left\{\left(r, H_t^p(q)\right)\colon\ t \in I\right\} \subset F^{-1}(V) \text{ for some } V \in \mathcal{V}\right\}.$$

Since H_t^p is $F_p^{-1}[\mathcal{V}]$-limited we have $\{p\} \times Q \subset V_p$ and since I is compact we have that V_p is open. Since Q is compact we can find, for each $p \in \mathbb{P}$, an open neighbourhood W_p of p such that $W_p \times Q \subset U_p \cap V_p$. By the 0-dimensionality of $\mathbb{P}$ we can select a discrete open covering $\mathcal{D}$ of $\mathbb{P}$ that refines $\{W_p\colon\ p \in \mathbb{P}\}$. Pick for each $D \in \mathcal{D}$ a $p(D) \in \mathbb{P}$ such that $D \subset W_{p(D)}$.

We now define the homotopy $\Gamma_t : \mathbb{P} \times Q \to \mathbb{P} \times Q$ by

$$\Gamma_t(r,q) = \left(r, H_t^{p(D)}(q)\right) \quad \text{if } (t,r,q) \in I \times \mathbb{P} \times Q \text{ with } r \in D \in \mathcal{D}.$$

By the construction we have: Γ_1 is a homeomorphism, $\Gamma_0 = 1_{\mathbb{P}\times Q}$, Γ_t is $F^{-1}[\mathcal{V}]$-limited, and the fibres of $F \circ \Gamma_1$ form a refinement of $\mathcal{U}$. So F is homotopically shrinkable and by Proposition 5 a homotopic near homeomorphism. Consequently, F is a fine homotopy equivalence and hence a hereditary shape equivalence. Since $F^{-1}(F(\mathbb{P} \times Z)) = \mathbb{P} \times Z$ the restriction $f = F|\mathbb{P} \times Z$ is a hereditary shape equivalence as well. $\square$

Lemma 8. *For any compactum Z there exists a hereditary shape equivalence $f : \mathbb{P}\times Z \to E$ such that for every compactum Y which is a hereditary shape equivalence image of Z there exists a $p \in \mathbb{P}$ such that $f(\{p\} \times Z)$ is homeomorphic to Y.*

Proof. Let $C(Q)$ denote the space of continuous maps of the Hilbert cube Q into itself endowed with the compact-open topology. Recall that $C(Q)$ is a completely metrizable space. We may assume that Z is a Z-set in Q. We show that the set

$$C_Z(Q) = \left\{f \in C(Q)\colon\ f^{-1}\left(f(Z)\right) = Z\right\}$$

is a $\mathcal{G}_\delta$-subset of $C(Q)$. Write $Q \setminus Z = \bigcup_{n=1}^\infty F_n$ with F_n compact and $F_n \subset F_{n+1}$ for $n \in \mathbb{N}$. Then $C_Z(Q) = \bigcap_{n=1}^\infty U_n$, where $U_n = \{\alpha \in C(Q)\colon\ \alpha(F_n) \cap \alpha(Z) = \emptyset\}$ is an open subset of $C(Q)$ for $n \in \mathbb{N}$. Let $\mathrm{CE}(Q)$ denote the subset of $C(Q)$ consisting of all cell-like maps of Q onto itself and let $\mathrm{CE}_Z(Q) = \mathrm{CE}(Q) \cap C_Z(Q)$. Note that $\mathrm{CE}(Q)$ is a closed subset of $C(Q)$ since it is the closure of the set of autohomeomorphisms of Q. So $\mathrm{CE}_Z(Q)$ is a $\mathcal{G}_\delta$-subset of the complete metric space $C(Q)$.

Hence there exists a continuous map $p \mapsto \nu_p$ from $\mathbb{P}$ onto $\mathrm{CE}_Z(Q)$. Note that since Q is compact and $\mathrm{CE}_Z(Q)$ is endowed with the compact-open topology the function $\nu_p(q)$ is continuous in $(p,q) \in \mathbb{P} \times Q$. Define the map $f:\mathbb{P} \times Z \to \mathbb{P} \times Q$ by

$$f(p,q) = \big(p, \nu_p(q)\big) \quad \text{for } p \in \mathbb{P} \text{ and } q \in Z.$$

Let $E = f(\mathbb{P} \times Z)$ and let $\pi:\mathbb{P} \times Q \to \mathbb{P}$ be the projection. If C is a compact subset of E then $f^{-1}(C)$ is contained in the compactum $\pi(C) \times Z$ so $f:\mathbb{P} \times Z \to E$ is a proper map. For every $p \in \mathbb{P}$, ν_p is a hereditary shape equivalence of Q with the property $\nu_p^{-1}(\nu_p(Z)) = Z$. Consequently, the restriction $\nu_p|Z$ is also a hereditary shape equivalence. Note that $\nu_p|Z$ is essentially identical to $f|\{p\} \times Z$ so that map is a hereditary shape equivalence as well for every $p \in \mathbb{P}$. By Lemma 7 the map $f:\mathbb{P} \times Z \to E$ is a hereditary shape equivalence. If $\zeta: Z \to Y$ is a hereditary shape equivalence of Z onto Y, then there exist a map $\alpha \in \mathrm{CE}_Z(Q)$ and a homeomorphism $h:\alpha(Z) \to Y$ such that $h \circ \alpha|Z = \zeta$. We have $\alpha = \nu_p$ for some $p \in \mathbb{P}$. Hence Y is homeomorphic to $\{p\} \times \nu_p(Z) = f(\{p\} \times Z)$. □

Proof of Theorem 1. Let α be a countable ordinal. By [14] there exists an α-dimensional completely metrizable space Z' which is universal for all α-dimensional compacta. Let Z be a countably dimensional compactification of Z' (see [10]). By Lemma 8 there exists a hereditary shape equivalence $f:\mathbb{P} \times Z \to E$ such that each hereditary shape equivalence image of Z is homeomorphic to $f(\{p\} \times Z)$ for some $p \in \mathbb{P}$. Moreover, if X is a compactum with $\operatorname{ind} X \leqslant \alpha$ and Y is a hereditary shape equivalence image of X then Y is homeomorphic to a subset of $f(\{p\} \times Z)$ for some $p \in \mathbb{P}$. Since $\mathbb{P} \times Z$ is obviously a countably dimensional complete space and E is also complete because f is proper the premise of the theorem guarantees that E is countably dimensional. Hence $\operatorname{ind} E$ is countable. We have

$$\sup\big\{\eta(X)\colon\ X \text{ is a compactum with } \operatorname{ind} X \leqslant \alpha\big\} \leqslant \operatorname{ind} E < \omega_1. \qquad \square$$

Conjecture 9. *If for some $\alpha < \omega_1$*

$$\sup\big\{\eta(X)\colon\ X \textit{ is a compactum with } \operatorname{ind} X \leqslant \alpha\big\} = \omega_1$$

then there exists a hereditary shape equivalence from a countably dimensional compactum onto a compactum that is not countably dimensional.

An idea for a proof is to start with Theorem 1 and then to construct a "compactification" of the hereditary shape equivalence $f:\mathbb{P} \times Z \to E$, which leads to the following:

Question 10. *If $h: X \to Y$ is a hereditary shape equivalence between complete spaces do there exist compactifications C and D of X respectively Y such that h extends to a hereditary shape equivalence $\overline{h}: C \to D$?*

It can be shown that if the answer to Question 10 is yes then Conjecture 9 is true.

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